Descriptive Set Theory
Lecture 10

A **game** on a set of moves A (this is usually ctbl, but not always) is: $a_i \in A$

P1: a_0 $\quad$ a_2 $\quad \dots$

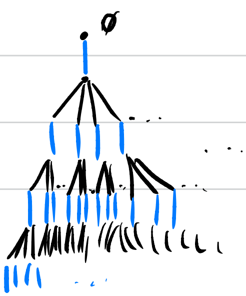
P2: $\quad$ a_1 $\quad$ a_3

This game has a **payoff** set $D \subseteq A^{\mathbb{N}}$ which determines who wins a given **run** (a_n) of the game, namely, P1 wins $:\Leftrightarrow (a_n) \in D$. We denote this game by $G_A(D)$ or simply $G(D)$ if A doesn't matter.

A **strategy** for P1 is a map $\varphi : A^{<\mathbb{N}} \to A$. We say that a run (a_n) of this game is played according to φ if $\forall k$, $a_{2k} = \varphi(a_1, a_2, \dots, a_{2k-1})$, where is $\emptyset$ for $k=0$.

Similarly, we define a strategy for P2.

More conveniently, we define a strategy as a tree on A as follows. A **strategy** for P1 is a tree T on A s.t.



(i) For each $(a_0, \dots, a_{2k}) \in T$, $\exists$ exactly one $a \in A$ s.t. $(a_0, \dots, a_{2k}, a) \in T$.

(ii) For each $(a_0, \dots, a_{2k-1}) \in T$, $\forall a \in A$, $(a_0, \dots, a_{2k-1}, a) \in T$.

Similarly, one defines a strategy for P2, swapping odd and even.
Given a strategy $\sigma \subseteq A^{<\mathbb{N}}$ (for one of the players), $x \in A^{\mathbb{N}}$ is a *run according to σ* if $x \in [\sigma]$.
A *winning* strategy for P1 (resp. P2) in the game $G(D)$, for $D \subseteq A^{\mathbb{N}}$, is a strategy σ s.t. $[\sigma] \subseteq D$ (resp. $[\sigma] \subseteq D^c$).
We say that the game $G(D)$ is *determined* if one of the players has a winning strategy.

Games with rules. A *game with rules* is a game with a tree T (on a set A) and a payoff set $D \subseteq [T]$ and P1 and P2 have to play moves, so that each *position* $(a_0, a_1, \dots, a_n)$ of the game is in T. Denote $G_T(D)$.

T

Note that such a game is *equivalent* to a game without rules by a slight modification of the payoff set.
Given $G_T(D)$, we turn this into $G_A(D')$, where $D' \subseteq A^{\mathbb{N}}$ is defined by: $\forall x \in A^{\mathbb{N}}$,
$$x \in D' :\Leftrightarrow (x \in D) \text{ or } (\exists n\ x|_n \notin T \text{ and the least such } n \text{ is odd}).$$

Note. D' is D union open, so unless D is closed, D and D' have the same complexity.

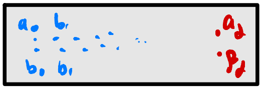
Nondetermined games. We use Axiom of Choice (AC) to build a nondetermined set $B \subseteq A^{\mathbb{N}}$ for any A of size >1.

Observation. If $G_A(D)$ is determined, then D or D^c contains a nonempty perfect set.

Proof. For any strategy σ, $[\sigma]$ is a nonempty perfect set. $\square$

Theorem (Bernstein, uses AC). For any ctbl A of size >1, $\exists$ a set $B \subseteq A^{\mathbb{N}}$ (called a **Bernstein set**) s.t. neither B nor B^c contains a nonempty perfect set. Hence, B is nondetermined.

Proof. By AC, we can well-order the set of all nonempty perfect sets, i.e. $\exists$ ordinal enumeration $(P_\alpha)_{\alpha<\lambda}$ of all such sets, where $\lambda \le 2^{\aleph_0}$ (continuum) is a cardinal. (**Remark.** $\lambda = 2^{\aleph_0}$, because a generic compact subset of $A^{\mathbb{N}}$ is perfect.) We recursively build a sequence $(a_\alpha, b_\alpha)_{\alpha<\lambda}$ s.t. $a_\alpha \overset{\neq}{,} b_\alpha \in P_\alpha$ and these haven't appeared before, i.e. $a_\alpha, b_\alpha \notin \{a_\gamma, b_\gamma : \gamma < \alpha\}$. Given $(a_\gamma, b_\gamma)_{\gamma<\alpha}$, the set $\{a_\gamma, b_\gamma : \gamma<\alpha\}$ is of cardinality $< \lambda \le 2^{\aleph_0}$ hence $P_\alpha \setminus \{a_\gamma, b_\gamma : \gamma < \alpha\}$



P_α

still has size continuum, so we can take $a_\alpha, b_\alpha \in P \setminus \{a_\gamma, b_\gamma : \gamma < \alpha\}$. Let $B := \{a_\alpha : \alpha < \lambda\}$ hence $B^c \supseteq \{b_\alpha : \alpha < \lambda\}$, hence $\forall \alpha$, $P_\alpha \not\subseteq B$ & $P_\alpha \not\subseteq B^c$. $\square$

Axiom of Determinacy (AD). For all A, all sets $D \subseteq A^{\mathbb{N}}$ are determined.

We just showed that AC $\Rightarrow$ not AD, but we believe that ZF + AD is consistent.

We'll show that all open sets are determined, also all closed sets are determined. It's a brilliant theorem of D. Martin that all Borel sets are determined. Determinacy of projections of Borel sets, i.e. analytic sets, is equivalent to ~~a~~ the existence of a measurable cardinal, and we can't even prove its consistency.

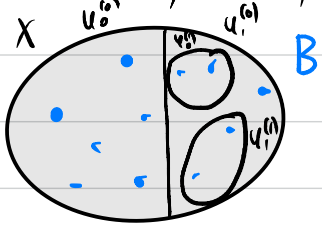
Perfect set property. A subset of a Polish space has the perfect set property (PSP) if it is either ctbl or contains a nonempty perfect set (hence a copy of $2^{\mathbb{N}}$).

Nonexample. A Bernstein set $B \subseteq 2^{\mathbb{N}}$ doesn't have PSP because by def, B doesn't contain a nonempty perfect set and it is not ctbl because o.w. B^c would be unctbl G_δ, hence Polish, so it would contain a $\emptyset \neq$ perfect set by Cantor – Bendixson.

The associated game. Let X (cake) be $\emptyset \neq$ perfect Polish space and let $\mathcal{U}$ be a ctbl basis. Fix a complete compatible metric d. For a given set B (blueberries) $\subseteq X$, the cut-and-choose game $G_X^{\blacksquare|\square}(B)$ is as follows:

P1:	$(U_0^{(0)}, U_1^{(0)})$	$(U_0^{(1)}, U_1^{(1)})$	$(U_0^{(2)}, U_1^{(2)})$	...
P2:		i_0	i_1	i_2

where $U_0^{(n)}, U_1^{(n)}$ are disjoint basic nonempty open sets (from $\mathcal{U}$) of vanishing diameter, i.e. $\operatorname{diam} U_*^{(n)} \to 0$, and $i_n \in \{0,1\}$. Moreover, $\overline{U_0^{(n+1)}}, \overline{U_1^{(n+1)}} \subseteq U_{i_n}^{(n)}$. P1 wins if $\bigcap_n U_{i_n}^{(n)} = \bigcap_n \overline{U_{i_n}^{(n)}} \subseteq B$. Note that $\bigcap_n U_{i_n}^{(n)} = \{x\}$ for some $x \in X$.



Theorem. (a) Player 1 has a winning strategy $\iff$ $B \supseteq$ perfect set $\neq \emptyset$.

(b) Player 2 has a winning strategy $\iff$ B is ctbl.

Proof. (a) $\Rightarrow$. Let σ be a winning strategy for Player 1. This σ defines a Cantor scheme in X, namely, $(U_s)_{s \in 2^{<\mathbb{N}}}$, where U_{1011} is the open set P2 chose after its moves $i_0 = 1$, $i_1 = 0$, $i_2 = 1$, $i_3 = 1$. This scheme has vanishing diameter and $\overline{U_{si}} \subseteq U_s$, and each is nonempty, so the domain of the associated map is the whole $2^{\mathbb{N}}$, hence $2^{\mathbb{N}} \hookrightarrow B$.

$\Leftarrow$. If B contains a $\overset{\emptyset_X}{\text{perfect}}$ set $P \subseteq B$, then P1 plays $U_0^{(0)}, U_1^{(1)}$ disjoint s.t. the both intersect P. And continues this way, which is possible because P is perfect. (Mimic the proof of the perfect set theorem, where we construct the Cantor scheme.) This gives a winning strategy for P1. $\boxed{a}$